V. Saxena [a,*], I. Kourakis [a], G. Sanchez-Arriaga [b], E. Siminos [c]

[a] Centre for Plasma Physics, School of Mathematics and Physics, Queen's University Belfast, Belfast BT7 1NN, Northern Ireland, United Kingdom
[b] Departamento de Física Aplicada, Escuela Técnica Superior de Ingenieros Aeronáuticos, Universidad Politécnica de Madrid, Madrid, Spain
[c] Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Str. 38, D-01187 Dresden, Germany

A B S T R A C T

Numerical investigations on mutual interactions between two spatially overlapping standing electromagnetic solitons in a cold unmagnetized plasma are reported. It is found that an initial state comprising of two overlapping standing solitons evolves into different end states, depending on the amplitudes of the two solitons and the phase difference between them. For small amplitude solitons with zero phase difference, we observe the formation of an oscillating bound state whose period depends on their initial separation. These results suggest the existence of a bound state made of two solitons in the relativistic cold plasma fluid model.

Keywords:
Laser plasma interaction
Fluid simulation
Soliton
Bound state

1. Introduction

The field of laser plasma interactions has seen a resurgence after the invention of Chirped Pulsed Amplification (CPA) technology, thanks to Mourou, Stickland and co-workers [1,2]. Modern lasers having ultra-high intensities ($I \geqslant 10^{18}$ W/cm^2) can drive electrons to relativistic quiver velocities. Interaction of such intense laser pulses with a plasma exhibits a rich variety of nonlinear phenomena. One such interesting phenomenon, namely the formation of electromagnetic solitons, has drawn much attention from plasma theorists and also from laser plasma experimentalists due to the intrinsic interest in their nonlinear properties as well as for their potential applications in areas like particle acceleration or in the fast ignition scheme of laser fusion. From theoretical point of view, a special class of exact one dimensional solitary wave solutions for modulated light pulses coupled to electron plasma waves in a relativistic cold plasma has received considerable numerical and analytic attention in the past few years [3–15]. Physically these solutions represent a nonlinear stationary state in which the light pulse is trapped in a plasma wave that it generates itself. It has been observed in numerical simulations [16,17] as well as in experiments [18,19], investigating the interaction of a circularly polarized laser pulse with a plasma that some part of the laser energy was trapped in non-propagating soliton like pulse structures.

In a laser plasma experiment, it is very likely that a large number of non-propagating localized solitary structures are excited which remain in the neighborhood of each other. These electromagnetic structures are expected to impart a force on each other to give rise to interesting phenomena like repulsion, merging of solitons or oscillatory bound states. In the present Letter, we investigate this issue, in detail, with the help of fluid simulations. In our simulations, we have followed the evolution of a pair of two circularly-polarized standing solitary wave solutions placed in the vicinity of each other with an emphasis on the effect of initial separation, amplitudes and relative phases of the two solutions. Of particular interest is the case of small amplitude in-phase solitary waves because they evolve to a state where the two solitary waves change their positions periodically in time. To the best of our knowledge this is the first time that a bound state consisting of two solitons is reported within the framework of the relativistic cold plasma fluid model. Previous works, mainly using PIC simulations, point out the merging of two s-polarized solitons [20] or their mutual repulsion in the case of out of phase solitons [21]. Our numerical results have a close resemblance to those analytically obtained by Gordon et al. [22] for the dynamical evolution of a two soliton solution, in the form of a bi-soliton state, of nonlinear Schrödinger equation which are relevant in nonlinear optics.

2. Governing equations, stationary solutions and fluid code

Our investigation relies on a one dimensional cold plasma fluid-Maxwell model which has been extensively used in the existing literature on laser plasma interactions. The ions are considered to

* Corresponding author.
E-mail address: v.saxena@qub.ac.uk (V. Saxena).

form a stationary neutralizing background for the cold electron fluid. The model equations are the electron continuity equation, the longitudinal electron momentum equation, Poisson's equation, and electromagnetic wave equation [5,14],

$$\frac{\partial n}{\partial t}+\frac{\partial (nu)}{\partial x}=0, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)(\gamma u)=\frac{\partial \phi}{\partial x}-\frac{1}{2\gamma}\frac{\partial A_\perp^2}{\partial x}, \tag{2}$$

$$\frac{\partial^2 \phi}{\partial x^2}=n-1, \tag{3}$$

$$\frac{\partial^2 \vec{A}_\perp}{\partial x^2}-\frac{\partial^2 \vec{A}_\perp}{\partial t^2}=\frac{n\vec{A}_\perp}{\gamma} \tag{4}$$

where $n(x)$ is the fluid density normalized to the background plasma density n_0, u is the longitudinal fluid velocity normalized to the speed of light c, $A_\perp(x)$ and $\phi(x)$ respectively represent the vector and electrostatic potential both normalized to m_ec^2/e. γ is the relativistic factor defined by $\gamma=\sqrt{1+p_\parallel^2+p_\perp^2}=\sqrt{(1+A_\perp^2)/(1-u^2)}$ where $p_\parallel=\gamma u$ and $p_\perp=A_\perp$ are the longitudinal and transverse components of the relativistic fluid momentum. The perpendicular electron momentum has been eliminated by using the relation $\vec{u}_\perp-\vec{A}_\perp/\gamma=0$, which follows from the conservation of the transverse canonical momentum. Time and length are respectively normalized to the inverse of the plasma frequency $\omega_{p0}=\sqrt{4\pi n_0e^2/m_e}$ and the corresponding skin depth c/ω_{p0}.

The coupled set of nonlinear equations (1)–(4) permits a variety of coherent stationary solutions. Among them, a class of traveling soliton like solutions have been discussed by several authors in the previous studies [4–12]. In the particular case of standing waves, Eqs. (1)–(4) admit an exact analytical solution [13]. The components of the vector potential transverse to the propagation direction x of a soliton are given by $A_y+iA_z=R_j(x)e^{i(\omega_jt+\theta_j)}$, where

$$R_j\equiv\frac{2\sqrt{1-\omega_j^2}\cosh[\sqrt{1-\omega_j^2}(x-x_j)]}{\cosh^2[\sqrt{1-\omega_j^2}(x-x_j)]+\omega_j^2-1} \tag{5}$$

where j indexes different solutions ($j=1,2$), ω_j is the frequency of the soliton and, for convenience, we introduced the position of the (peak of the) soliton x_j and the phase θ_j, which are arbitrary in the case of a single soliton state. The corresponding fluid velocity is $u_j=0$ and the density expression is

$$n_j=1.0+\frac{(1-\omega_j^2)^2[\cosh 4\xi_j-2(2\omega_j^2-1)\cosh 2\xi_j-3]}{(\cosh^2\xi_j+\omega_j^2-1)^3} \tag{6}$$

where $\xi_j=\sqrt{1-\omega_j^2}(x-x_j)$ and the electrostatic potential can be easily calculated by solving the Poisson equation (3). The maximum amplitude $R_{0,j}\equiv R_j(x_j)$ is given by

$$R_{0,j}=\frac{2\sqrt{1-\omega_j^2}}{\omega_j^2}. \tag{7}$$

The amplitude is zero for $\omega_j=1$ while it takes its maximum value $R_{0,j}=\sqrt{3}$ for $\omega_j=\omega_{cr}=\sqrt{2/3}$. For $\omega<\omega_{cr}$ the density becomes negative and the solitons cease to exist.

The set of Eqs. (1)–(4) are solved using the LCPFCT package of subroutines which are based on the flux correction algorithm proposed by Boris et al. [23]. The subroutine package is capable of solving hyperbolic partial differential equations of continuity/convective form such as (1)–(3). Fortunately, the wave equation (4) can also be cast into four equations having convective/continuity form and can thus be solved using LCPFCT. The Courant condition for the numerical stability of the algorithm has been taken care of while choosing the grid spacing and appropriate time step. Periodic boundary conditions in x direction are also imposed.

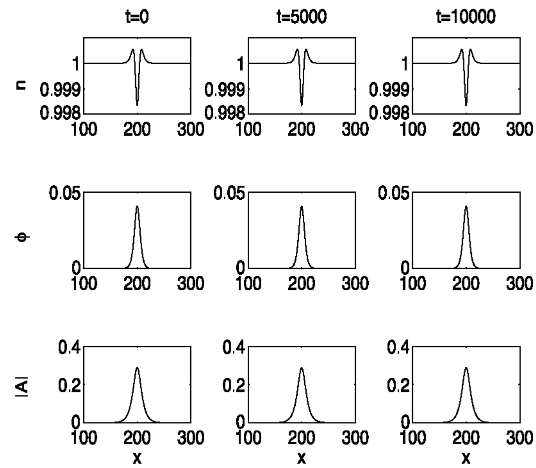

Fig. 1. The spatial profiles of density n, electrostatic potential ϕ and vector potential $|A|$ at three different time instants for a single peak nonlinear standing wave solutions corresponding to $\omega=0.99$ for the coupled laser plasma system.

The fluid code has been validated by using solution (5), which is known to exhibit excellent stability properties [13]. We chose an exemplary initial condition with $\omega=0.99$, corresponding to a vector potential amplitude of $R_0=0.2879$, and we integrated the equations forward in time. The field profiles at different time instants are shown in Fig. 1. The soliton solution remains unchanged for the entire simulation time ($\sim 10^4\omega_{p0}^{-1}$) and thus validates our code.

3. Interaction of two standing solitons

We now present our simulation results regarding the interaction between two solitary wave solutions. In order to investigate such a physical scenario, we have initialized our fluid code with a profile made of the sum of two Esirkepov's solitons. For instance the components of the vector potential are $A_y+iA_z=R_1(x)e^{i(\omega_1t+\theta_1)}+R_2(x)e^{i(\omega_2t+\theta_2)}$ and, similarly, the remaining fluid and electromagnetic variables are initialized as linear superpositions of two single soliton solutions. As compared with other numerical works in which multiple solitons are excited by a laser pulse, our approach singles out the effect related to the interaction of two solitons, by-passing complications due to laser plasma interactions. Obviously, due to nonlinearity, our linear combination of Esirkepov's solitons is not a solution of Eqs. (1)–(4) and the system evolves to different states depending on the parameters controlling the two solitons. Such parameters are (i) the two frequencies ω_1 and ω_2 (or amplitudes), (ii) their mutual separation $d\equiv x_1-x_2$, and (iii) their relative phase $\Delta\theta\equiv\theta_1-\theta_2$. In the following sub-sections we discuss the effect of all these parameters on the evolution of the two solitons.

Case I: Small and similar amplitudes with no phase difference. The first set of simulations were run with an initial condition of parameter values $\omega_1=\omega_2=0.999$, $R_{0,1}=R_{0,2}=0.0896$, $\Delta\theta=0$. As can be seen in Fig. 2, where the initial distance between solitons is equal to $d=100c/\omega_{p0}$, the two solitons start moving towards each other. At time $\omega_{p0}t=3400$ they merge into one structure from which they emerge again and the initial state of the system is recovered at time $\omega_{p0}t=6800$. This cycle continues throughout our

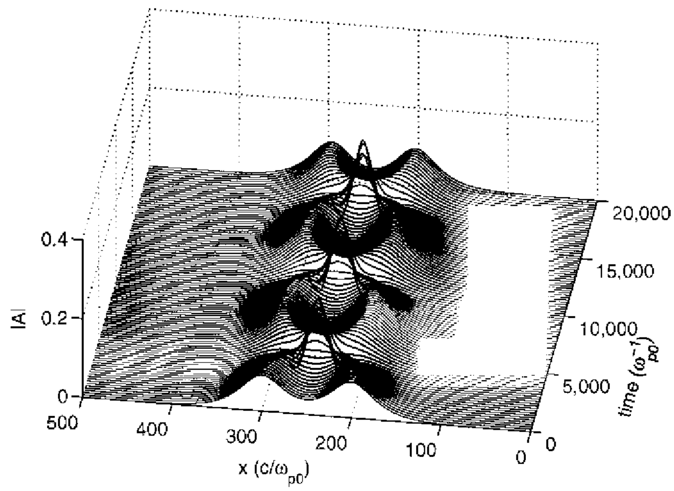

Fig. 2. The temporal evolution of a pair of single peak nonlinear standing solitary solutions corresponding to $\omega = 0.999$ and initial distance $d = 100c/\omega_{p0}$.

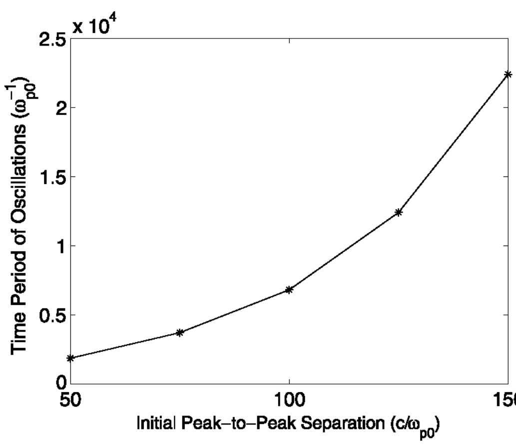

Fig. 3. Oscillation period versus initial peak-to-peak separation between two solitary solutions each with $\omega = 0.999$.

numerical simulations, in fact for more than 10^4 plasma periods. Another interesting feature is that, even though the two solitons are exactly equal at $t = 0$, the amplitudes become slightly different after the collision (for instance amplitudes 0.09636 and 0.09631 at $\omega_{p0}t = 2 \times 10^4$).

Simulations with the same parameters but different distances between solitons show that, for attraction to occur, a significant part of the body of the two solitons should spatially overlap. Moreover, the time period of oscillation directly depends on the initial separation between the peaks of the two solitons, i.e., the farther apart the two solitons' centres (peaks) are initially from each other, the slower they oscillate around the mean position of the pair. This period versus initial distance dependence can be seen in Fig. 3.

Case II: Small but different amplitudes with no phase difference. When the amplitudes of two solitons are chosen to be small but different, a very interesting effect is observed in the simulations. It is found that the two structures continue to remain bounded but with a finite separation unlike in the equal amplitude case discussed above where the two structures propagate towards each other until they meet at their mean position. Thus, the interaction between the two solitary structures can be controlled by changing the amplitude difference between them. This is an important feature which has an analogy in optical fiber communication where it is utilized for lossless transmission of signal [24]. A numerical simulation is shown in Fig. 4 where two non-propagating soliton solutions with

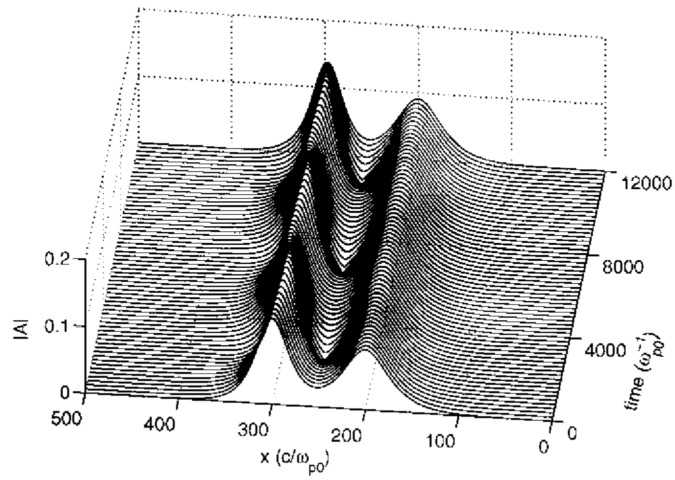

Fig. 4. The temporal evolution of a pair of single peak nonlinear standing solitary solutions corresponding to $\omega_1 = 0.999$ and $\omega_2 = 0.998$.

$\omega_1 = 0.999$ ($R_{0,1} = 0.0896$), $\omega_2 = 0.998$ ($R_{0,2} = 0.1269$), $\Delta\theta = 0$ and $d = 100c/\omega_{p0}$ are chosen for the purpose of presentation. It can be noticed that the two solitary structures never come closer than a finite separation but rather oscillate between states of minimum and maximum separation. A small oscillation in the amplitude of the solution with $\omega_2 = 0.998$ is also observed with $R_{0,2}$ taking values between ≈ 0.13 and ≈ 0.154.

Case III: Small and similar amplitudes with finite phase difference. If a phase difference between the two solitary solutions is introduced, bound states are not always observed. The top panel of Fig. 5 shows the spatio-temporal evolution of the vector potential for a simulation started with two solitons with $\omega_1 = \omega_2 = 0.999$ ($R_{0,1} = R_{0,2} = 0.0896$), phase difference $\Delta\theta = 0.1\pi$ and distance $d = 100c/\omega_{p0}$. It can be seen that the two structures are first attracted towards each other but eventually they separate and evolve into escaping solitons. If we choose a smaller phase difference the solitons complete some (almost periodic) oscillations before they escape, *e.g.* for $\Delta\theta = 10^{-2}\pi$ they complete two oscillations, see bottom panel of Fig. 5. In general, for decreasing $\Delta\theta$ the solitons oscillate for a longer time before they separate. For $\Delta\theta = 10^{-4}\pi$ the solitons appear to converge to a bound state, as we observe no separation after 7 oscillation periods (not shown).

Case IV: Large amplitude regime. Now we consider an initial condition made of two standing solitary waves with parameters $\omega_1 = \omega_2 = 0.99$ ($R_{0,1} = R_{0,2} = 0.2878$), $\Delta\theta = 0$ and $d = 60c/\omega_{p0}$. In Fig. 6 we observe that the two structures are attracted towards each other as in the small amplitude case and completely overlap with each other. However, unlike in the small amplitude case (Case I), the two structures do not remain intact after the collision and undergo a change in shape as well as in amplitudes. Moreover, we note that the emerging structures keep moving in the opposite direction and thus the final state in this case is a pair of escaping solitons.

To summarize, we have reported numerical observations regarding the time evolution of two closely spaced standing electromagnetic solitary waves in a plasma. Cases I and II, which correspond to small amplitude solitons with no phase difference, are of special interest as they show, for the first time, that bound states consisting of two solitons oscillating periodically in time can occur within the relativistic cold fluid framework. Case III, indicates that the introduction of a large enough phase difference between the two solitons can lead to an unbounded solution, consisting of escaping solitons, after an initial oscillatory phase.

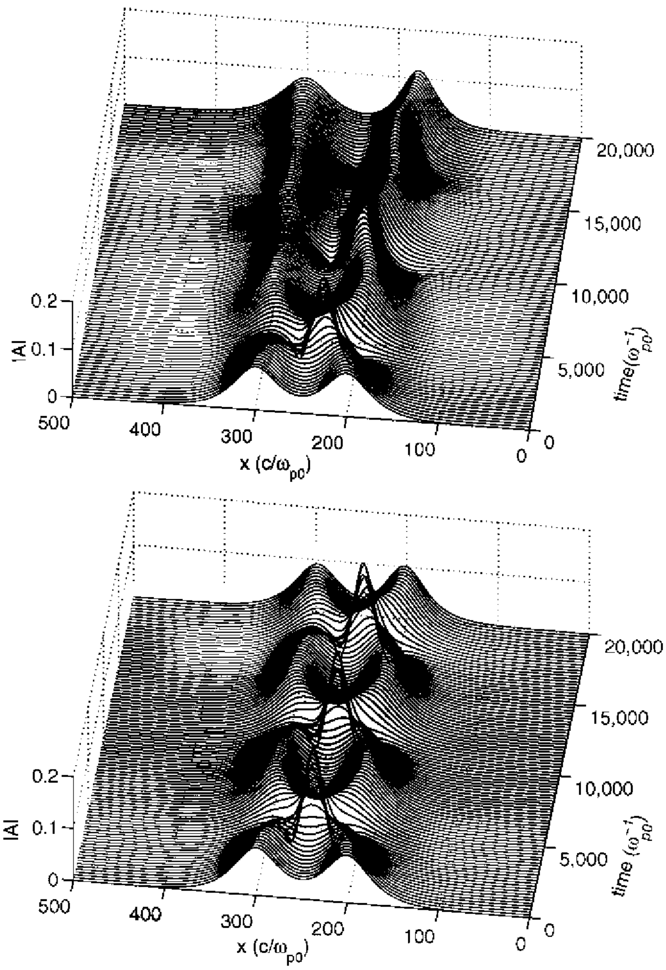

Fig. 5. Top panel: the temporal evolution of a pair of single peak nonlinear standing solitary solutions corresponding to $\omega_1 = \omega_2 = 0.999$ but with a finite phase difference $\Delta\theta = 0.1\pi$. Bottom panel: the same as in the top panel, but with $\Delta\theta = 10^{-2}\pi$.

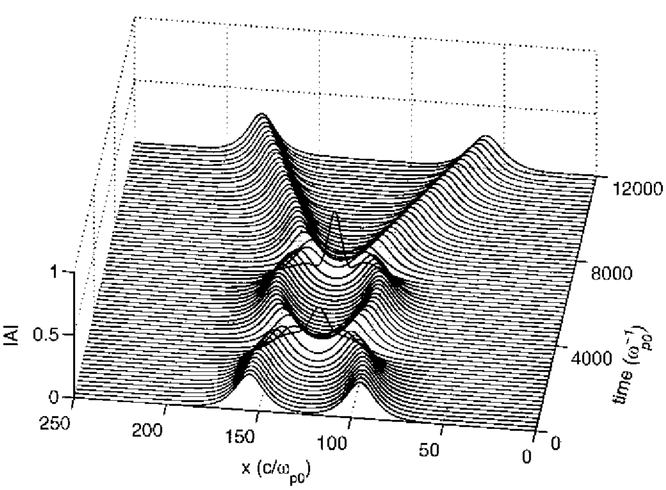

Fig. 6. The temporal evolution of a pair of single peak nonlinear standing solitary solutions corresponding to $\omega = 0.99$ and with no phase difference $\Delta\theta = 0$.

In the case of two large amplitude solitons (Case IV), the initial oscillatory phase hints to the existence of a bound state. However, the initial condition used here is not within its basin of attraction and the two solitons eventually diverge. In order to determine initial conditions which would lead to oscillations involving large amplitude solitons, the solutions presented here would have to be refined by iterative methods such as those used, for instance, in Ref. [12] for s-polarized solitons.

It is well known that in small amplitude limit the solitary waves are described by the nonlinear Schrödinger equation (NLSE). We recall that the NLSE is an integrable equation that admits multi-soliton solutions. Since our fluid simulations relied on a superposition of two standing solitons, of special interest appears to be the case of NLSE solutions made of two pulses with exactly equal velocities. These solutions are called bi-solitons [25] and they are bound states with time periodic behavior, similar to the behavior observed in our simulations. The NLSE thus seems to capture the essential aspects of the dynamics, and provides an appropriate model to be used as a basis in a detailed investigation of soliton interaction. This exceeds our scope here, and will be the object of a future work.

Our numerical observations, for instance the dependence of the interaction on the distance between the solitons or the existence of solitons with different amplitudes after the collision, are in qualitative agreement with the analytical and numerical results of Xu et al. [26], who investigated the interaction between soliton solutions of the NLSE with higher-order effects. Similar studies of interacting pulses have been carried out in nonlinear optics, relying on the NLSE paradigm [22]. We emphasise the fact that our investigation at hand has focused on interacting Esirkepov-type solutions, as given by expression (5) above [13]. Therefore, our result apply to finite (large) amplitude electromagnetic pulse excitations, in contrast to the small amplitude NLSE theory (which does seem to capture the basic qualitative aspects of the dynamics, yet is essentially distinct in nature, and limits itself to the small amplitude limit).

In conclusion, we have reported a series of numerical observations of the interaction of two closely located standing solitary waves in a cold fluid plasma model. Such a configuration is studied in a laser plasma framework for the first time, to our best knowledge. The effect of initial spatial separation, initial phase difference and of amplitude difference between the constituent solitons on the interaction characteristics has been investigated. The numerical observations presented here provide an analogy with results available in other fields, for example in nonlinear optics [22]. Our findings advance current knowledge in nonlinear laser plasma interactions, providing inspiration for further studies. Furthermore, these results should be of importance to other fields where nonlinear dynamics are relevant.

Acknowledgements

We would like to thank Mickael Grech for critical reading of the manuscript. VS would like to acknowledge Amita Das (Institute for Plasma Research, Gandhinagar, India) for her valuable suggestion on the usage of LCPFCT subroutines to solve the electromagnetic wave equation. Financial support from the UK EPSRC (Engineering and Physical Science Research Council) via grant EP/I031766/1 is gratefully acknowledged.

References

[1] D. Strickland, G. Mourou, Opt. Commun. 56 (1985) 219.
[2] G. Mourou, D. Umstadter, Phys. Fluids B 4 (1992) 2315.
[3] A.I. Akhiezer, R.V. Polovin, Sov. Phys. JETP 3 (1956) 696.
[4] V.A. Kozlov, A.G. Litvak, E.V. Suvorov, Sov. Phys. JETP 49 (1979) 75.
[5] P.K. Kaw, A. Sen, T. Katsouleas, Phys. Rev. Lett. 68 (1992) 3172.
[6] R.N. Sudan, Y.S. Dimant, O.B. Shiryaev, Phys. Plasmas 4 (1997) 1489.
[7] D. Farina, S.V. Bulanov, Phys. Rev. Lett. 86 (2001) 5289.
[8] M. Lontano, M. Passoni, S.V. Bulanov, Phys. Plasmas 10 (2003) 639.
[9] D. Farina, S.V. Bulanov, Plasma Phys. Control. Fusion 47 (2005) A73.
[10] S. Sundar, A. Das, V. Saxena, P.K. Kaw, A. Sen, Phys. Plasmas 18 (2011) 112112.
[11] G. Sanchez-Arriaga, E. Siminos, E. Lefebvre, Plasma Phys. Control. Fusion 53 (2011) 045011.
[12] G. Sanchez-Arriaga, E. Siminos, E. Lefebvre, Phys. Plasmas 18 (2011) 082304.

[13] T.Zh. Esirkepov, F.F. Kamenets, S.V. Bulanov, N.M. Naumova, JETP Lett. 68 (1998) 36.
[14] V. Saxena, A. Das, A. Sen, P. Kaw, Phys. Plasmas 13 (2006) 032309.
[15] G. Sanchez-Arriaga, E. Lefebvre, Phys. Rev. E 84 (2011) 036403.
[16] S.V. Bulanov, N.M. Naumova, F. Pegoraro, Phys. Plasmas 1 (1994) 745.
[17] M. Tushentov, A. Kim, F. Cattani, D. Anderson, M. Lisak, Phys. Rev. Lett. 87 (2001) 275002.
[18] M. Borghesi, S. Bulanov, D.H. Campbell, et al., Phys. Rev. Lett. 88 (2002) 135002.
[19] M. Borghesi, D.H. Campbell, A. Schiavi, et al., Phys. Plasmas 9 (2002) 2214.
[20] F. Pegoraro, S.V. Bulanov, F. Califano, et al., Phys. Scr. T 84 (2000) 89.
[21] G. Sanchez-Arriaga, E. Lefebvre, Phys. Rev. E 84 (2011) 036404.
[22] J.P. Gordon, Opt. Lett. 8 (1983) 596.
[23] J.P. Boris, D.L. Book, Methods Comput. Phys. 16 (1976) 85.
[24] P.L. Chu, C. Desem, Electron. Lett. 21 (1985) 1134.
[25] V.E. Zakharov, B.A. Shabat, Sov. Phys. JETP 34 (1972) 62.
[26] Z. Xu, Lu Li, Z. Li, G. Zhou, Opt. Commun. 210 (2002) 375.